NonCommutative Rings and their Applications, VII

Université d'Artois, Faculté des Sciences Jean Perrin de Lens

5th–7th July 2021, Lens (France)

Trusses

Małgorzata Elżbieta Hryniewicka

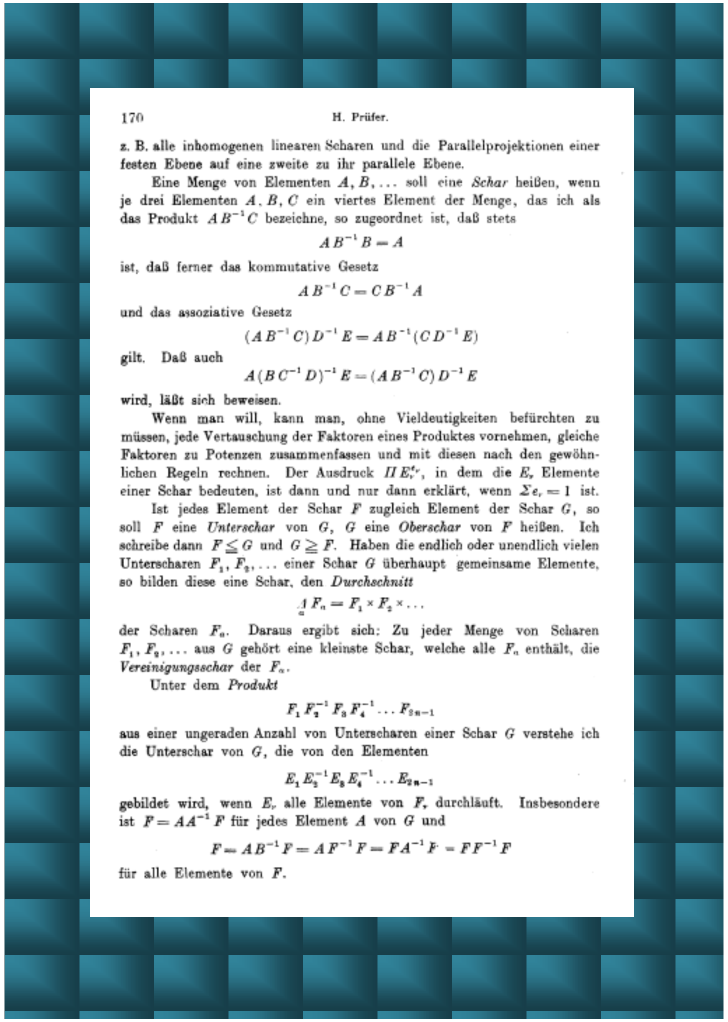
Faculty of Mathematics, University of Białystok

Ciołkowskiego 1M, 15-245 Białystok, Poland

e-mail: margitt@math.uwb.edu.pl; mhryniewicka@uwb.edu.pl

Heinz Prüfer [*Theorie der Abelschen Gruppen. I. Grundeigenschaften*, Mathematische Zeitschrift 20 (1924), 165-187, page 170].

Reinhold Baer [*Zur Einführung des Scharbegriffs*, Journal für die Reine und Angewandte Mathematik 160 (1929), 199-207, page 202].



170 H. Prüfer.

z. B. alle inhomogenen linearen Scharen und die Parallelprojektionen einer festen Ebene auf eine zweite zu ihr parallele Ebene.

Eine Menge von Elementen $A, B, \ldots$ soll eine *Schar* heißen, wenn je drei Elementen A, B, C ein viertes Element der Menge, das ich als das Produkt $AB^{-1}C$ bezeichne, so zugeordnet ist, daß stets

$$AB^{-1}B = A$$

ist, daß ferner das kommutative Gesetz

$$AB^{-1}C = CB^{-1}A$$

und das assoziative Gesetz

$$(AB^{-1}C)D^{-1}E = AB^{-1}(CD^{-1}E)$$

gilt. Daß auch

$$A(BC^{-1}D)^{-1}E = (AB^{-1}C)D^{-1}E$$

wird, läßt sich beweisen.

Wenn man will, kann man, ohne Vieldeutigkeiten befürchten zu müssen, jede Vertauschung der Faktoren eines Produktes vornehmen, gleiche Faktoren zu Potenzen zusammenfassen und mit diesen nach den gewöhnlichen Regeln rechnen. Der Ausdruck $\Pi E_\nu^{e_\nu}$, in dem die E_ν Elemente einer Schar bedeuten, ist dann und nur dann erklärt, wenn $\Sigma e_\nu = 1$ ist.

Ist jedes Element der Schar F zugleich Element der Schar G, so soll F eine *Unterschar* von G, G eine *Oberschar* von F heißen. Ich schreibe dann $F \leqq G$ und $G \geqq F$. Haben die endlich oder unendlich vielen Unterscharen $F_1, F_2, \ldots$ einer Schar G überhaupt gemeinsame Elemente, so bilden diese eine Schar, den *Durchschnitt*

$$\underset{\alpha}{\Delta} F_\alpha = F_1 \times F_2 \times \ldots$$

der Scharen F_α. Daraus ergibt sich: Zu jeder Menge von Scharen $F_1, F_2, \ldots$ aus G gehört eine kleinste Schar, welche alle F_α enthält, die *Vereinigungsschar* der F_α.

Unter dem *Produkt*

$$F_1 F_2^{-1} F_3 F_4^{-1} \ldots F_{2n-1}$$

aus einer ungeraden Anzahl von Unterscharen einer Schar G verstehe ich die Unterschar von G, die von den Elementen

$$E_1 E_2^{-1} E_3 E_4^{-1} \ldots E_{2n-1}$$

gebildet wird, wenn E_ν alle Elemente von F_ν durchläuft. Insbesondere ist $F = AA^{-1}F$ für jedes Element A von G und

$$F = AB^{-1}F = AF^{-1}F = FA^{-1}F = FF^{-1}F$$

für alle Elemente von F.

202 *Baer, Zur Einführung des Scharbegriffs.*

Setzen wir hier (5) ein, so ergibt sich:

$$(\mathfrak{z}_1 \mathfrak{x}_{i_1} \mathfrak{z}_2)^{e_2} (\mathfrak{z}_1 \mathfrak{x}_{i_1} \mathfrak{z}_2)^{e_3} J''(\cdot) = \mathfrak{z}_2^{-1} \mathfrak{x}_{i_1}^{e_2} \mathfrak{x}_{i_1}^{e_3} J''(\cdot)\, \mathfrak{z}_2$$

und hieraus folgt wie oben sukzessive:

$$e_2 = -1, \; e_3 = +1,$$

so daß sich:

$$J(\cdot) = \mathfrak{x}_{i_1} \mathfrak{x}_{i_2}^{-1} \mathfrak{x}_{i_3} J''(\cdot)$$

ergibt.

Analog weiter schließend erhält man:

(6) $$J(\cdot) = \mathfrak{x}_{i_1} \mathfrak{x}_{i_2}^{-1} \mathfrak{x}_{i_3} \mathfrak{x}_{i_4}^{-1} \mathfrak{x}_{i_5} \cdots \mathfrak{x}_{i_{m-1}}^{-1} \mathfrak{x}_{i_m},$$

wo stets $i_k \neq i_{k+1}$ und notwendig $m > 1$ und ungerade ist.

Sei $m = 2l + 1$, $l \geqq 1$.

Dann ist unser Satz bereits für $l = 1$ bewiesen; sei er auch für $l - 1$ bewiesen; es ist $J(\cdot) = \bar{J}(\cdot)\, \mathfrak{x}_{i_{m-1}}^{-1} \mathfrak{x}_{i_m}$, wo also n. V. $\bar{J}$ bereits rational und durch Einsetzen aus $F(\cdot)$ aufgebaut ist; dann ist:

$$J(\cdot) = F[\bar{J}(\cdot), \mathfrak{x}_{i_{m-1}}, \mathfrak{x}_{i_m}],$$

womit unser Satz bewiesen ist.

Bemerkung: Daß $F(\cdot)$ in speziellen Fällen kein vollständiges Invariantensystem ist, zeigt das folgende Beispiel:

$\mathfrak{G}$ sei *Abelsch* und werde durch $\mathfrak{a}_i$ mit $\mathfrak{a}_i^2 = 1$ $(i = 1, 2, 3, 4, 5)$ erzeugt. Dann wird auch

$$J[(\mathfrak{x}_i)] = \mathfrak{x}_1 \mathfrak{x}_2 \mathfrak{x}_3 \mathfrak{x}_4$$

durch A isomorph abgebildet; denn die allgemeine ähnliche Abbildung ist:

$$\mathfrak{x} \to \alpha(\mathfrak{x}) = \mathfrak{x}\mathfrak{z};$$

also ist:

$$J[(\alpha(\mathfrak{x}_i))] = \mathfrak{x}_1 \mathfrak{x}_2 \mathfrak{x}_3 \mathfrak{x}_4 \mathfrak{z}^4 = \mathfrak{x}_1 \mathfrak{x}_2 \mathfrak{x}_3 \mathfrak{x}_4 \mathfrak{z} = \alpha[J((\mathfrak{x}_i))].$$

und $J(\cdot)$ hat nicht die Form (6). Übrigens gilt unser Satz 2 auch für *Abelsche* Gruppen mit Elementen unendlich hoher Ordnung.

§ 2. Die Scharen.

Definition 2: *Eine Teilmenge $\mathfrak{S}$ der Gruppe $\mathfrak{G}$ heißt dann und nur dann eine Schar, wenn $\mathfrak{S}$ mit den Elementen $\mathfrak{a}_i (i = 1, 2, 3)$ stets auch das Element $\mathfrak{a}_1 \mathfrak{a}_2^{-1} \mathfrak{a}_3$ enthält.*

Satz 3: 1. *Eine Schar $\mathfrak{S}$ in $\mathfrak{G}$ ist dann und nur dann eine Untergruppe von $\mathfrak{G}$, wenn $\mathfrak{S}$ die Identität von $\mathfrak{G}$ enthält* [1]).

2. *Eine die Identität enthaltende Teilmenge $\mathfrak{S}$ von $\mathfrak{G}$ ist dann und nur dann eine Schar, wenn sie eine Untergruppe von $\mathfrak{G}$ ist.*

3. *Eine Teilmenge $\mathfrak{S}$ von $\mathfrak{G}$ ist dann und nur dann eine Schar in $\mathfrak{G}$, wenn es eine Untergruppe $\mathfrak{S}_r$ bezw. $\mathfrak{S}_l$ von $\mathfrak{G}$ gibt, so daß $\mathfrak{S}$ rechts- bezw. linksseitige Restklasse nach $\mathfrak{S}_r$ bezw. $\mathfrak{S}_l$ ist. $\mathfrak{S}_r$ bezw. $\mathfrak{S}_l$ sind hierdurch und durch $\mathfrak{S}$ eindeutig bestimmt.*

Beweis: I. Eine Teilmenge $\mathfrak{S}$ von $\mathfrak{G}$ ist dann und nur dann eine Untergruppe von $\mathfrak{G}$, wenn sie mit $\mathfrak{a}$ und $\mathfrak{b}$ auch $\mathfrak{a}^{-1}$ und $\mathfrak{a} \cdot \mathfrak{b}$ enthält; sie enthält also notwendig $\mathfrak{a}\mathfrak{a}^{-1} = 1$ und mit $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ auch $\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}$.

[1]) Führt man den Scharbegriff abstrakt ein, wie es *Prüfer*, l. c. p. 170 tut, indem man unter einer Schar ein Elementesystem versteht, in dem je drei Elemente $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ eindeutig ein viertes $\mathfrak{a}\,\mathfrak{b}^{-1}\,\mathfrak{c}$ bestimmen, so daß I. $\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{b} = \mathfrak{b}\mathfrak{b}^{-1}\mathfrak{a} = \mathfrak{a}$, II. $(\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c})\,\mathfrak{b}^{-1}\mathfrak{e} = \mathfrak{a}\,(\mathfrak{b}\mathfrak{c}^{-1}\mathfrak{b})^{-1}\mathfrak{e} = \mathfrak{a}\mathfrak{b}^{-1}(\mathfrak{c}\mathfrak{b}^{-1}\mathfrak{e}) = \mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}\mathfrak{b}^{-1}\mathfrak{e}$, III. die Gl. $\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c} = \mathfrak{b}$ nach jedem der Elemente $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ auf nur eine Weise auflösbar ist — die Lösbarkeit überhaupt läßt sich beweisen —, so kann man von der Schar jederzeit zur Gruppe übergehen, indem man irgendein festes Element $\mathfrak{o}$ als Einheit auszeichnet und als Gruppenkomposition $\mathfrak{a} \circ \mathfrak{b} = \mathfrak{a}\mathfrak{o}^{-1}\mathfrak{b}$ einführt.

A heap is an algebraic system $(H, [-,-,-])$ consisting of a nonempty set H, and a ternary operation

$$[-,-,-]\colon H \times H \times H \to H, \quad (x,y,z) \mapsto [x,y,z]$$

satisfying

the heap associativity $[[x,y,z],t,u] = [x,y,[z,t,u]]$, $\neq [x,[y,z,t],u]$

Mal'cev identities $[x,x,y] = y = [y,x,x]$, $\neq [x,y,x]$

where $x,y,z,t,u \in H$. A heap $(H,[-,-,-])$ is abelian, if satisfies

the heap commutativity $[x,y,z] = [z,y,x]$,

where $x,y,z \in H$.

A heap homomorphism is a function $\varphi\colon (H,[-,-,-,]) \to (\widetilde{H},[-,-,-,])$ respecting the heap operations

$$\varphi([x,y,z]) = [\varphi(x),\varphi(y),\varphi(z)],$$

where $x,y,z \in H$.

Theorem. Given a group $(G, \circ, 1)$, let

$$[-,-,-]_\circ \colon G \times G \times G \to G, \quad [x,y,z]_\circ := x \circ y^{-1} \circ z,$$

where $x, y, z \in G$. Then

(a) $(G, [-,-,-]_\circ)$ is a heap.

Indeed, for any $x, y, z, t, u \in G$,

$$[[x,y,z]_\circ, t, u]_\circ = (x \circ y^{-1} \circ z) \circ t^{-1} \circ u = x \circ y^{-1} \circ (z \circ t^{-1} \circ u) = [x, y, [z,t,u]_\circ]_\circ$$

$$[x,x,y]_\circ = x \circ x^{-1} \circ y = y = y \circ x^{-1} \circ x = [y,x,x]_\circ.$$

(b) If $(G, \circ, 1)$ is an abelian group, then $(G, [-,-,-]_\circ)$ is an abelian heap.

Indeed, for any $x, y, z \in G$,

$$[x,y,z]_\circ = x \circ y^{-1} \circ z = z \circ y^{-1} \circ x = [z,y,x]_\circ.$$

(c) Every group homomorphism $\varphi \colon (G, \circ, 1) \to (\widetilde{G}, \circ, 1)$

is an associated heap homomorphism $\varphi \colon (G, [-,-,-]_\circ) \to (\widetilde{G}, [-,-,-]_\circ)$.

Indeed, for any $x, y, z \in G$,

$$\varphi([x,y,z]_\circ) = \varphi(x \circ y^{-1} \circ z) = \varphi(x) \circ \varphi(y)^{-1} \circ \varphi(z) = [\varphi(x), \varphi(y), \varphi(z)]_\circ. \qquad \square$$

Theorem. Given a heap $(H, [-,-,-])$ and $e \in H$, let

$$\circ_e \colon H \times H \to H, \qquad x \circ_e y := [x, e, y],$$

where $x, y \in H$. Then

(a) $(H, \circ_e, e)$ is a group, known as a retract of $(H, [-,-,-])$.

Indeed, for any $x, y, z \in H$,

$(x \circ_e y) \circ_e z = [[x, e, y], e, z] = [x, e, [y, e, z]] = x \circ_e (y \circ_e z)$

$e \circ_e x = [e, e, x] = x = [x, e, e] = x \circ_e e$

$[e, x, e] \circ_e x = [[e, x, e], e, x] = [e, x, [e, e, x]] = [e, x, x] = e =$

$= [x, x, e] = [[x, e, e], x, e] = [x, e, [e, x, e]] = x \circ_e [e, x, e]$,

so $x^{-1} = [e, x, e]$.

(b) If $(H, [-,-,-])$ is an abelian heap, then $(H, \circ_e, e)$ is an abelian group.

Indeed, for any $x, y \in H$,

$x \circ_e y = [x, e, y] = [y, e, x] = y \circ_e x$.






(c) If $\varphi\colon (H,[-,-,-]) \to (\widetilde{H},[-,-,-])$ is a heap homomorphism,

then for any $e \in H$, $\tilde{e} \in \widetilde{H}$, the functions

$$\widehat{\varphi}\colon (H, \circ_e, e) \to (\widetilde{H}, \circ_{\tilde{e}}, \tilde{e}), \qquad x \mapsto [\varphi(x), \varphi(e), \tilde{e}]$$

$$\widehat{\varphi}^{\circ}\colon (H, \circ_e, e) \to (\widetilde{H}, \circ_{\tilde{e}}, \tilde{e}), \qquad x \mapsto [\tilde{e}, \varphi(e), \varphi(x)]$$

are associated group homomorphisms.

Indeed, for any $x, y \in H$,

$$\begin{aligned}
\widehat{\varphi}(x \circ_e y) &= [\varphi(x \circ_e y), \varphi(e), \tilde{e}] = [\varphi([x,e,y]), \varphi(e), \tilde{e}] = \\
&= [[\varphi(x), \varphi(e), \varphi(y)], \varphi(e), \tilde{e}] = [\varphi(x), \varphi(e), [\varphi(y), \varphi(e), \tilde{e}]] = \\
&= [\varphi(x), \varphi(e), \widehat{\varphi}(y)] = [\varphi(x), \varphi(e), [\tilde{e}, \tilde{e}, \widehat{\varphi}(y)]] = \\
&= [[\varphi(x), \varphi(e), \tilde{e}], \tilde{e}, \widehat{\varphi}(y)] = [\widehat{\varphi}(x), \tilde{e}, \widehat{\varphi}(y)] = \widehat{\varphi}(x) \circ_{\tilde{e}} \widehat{\varphi}(y).
\end{aligned}$$

In a similar manner, $\widehat{\varphi}^{\circ}(x \circ_e y) = \widehat{\varphi}^{\circ}(x) \circ_{\tilde{e}} \widehat{\varphi}^{\circ}(y)$. □




A group $(G, \circ, 1)$

$\Downarrow$

The heap $(G, [-,-,-]_\circ)$ associated to the group $(G, \circ, 1)$,

where $[x, y, z]_\circ := x \circ y^{-1} \circ z$

$\Downarrow$

$\forall e \in G$, The group $(G, \circ_e, e)$ associated to the heap $(G, [-,-,-]_\circ)$,

where $x \circ_e y := [x, e, y]_\circ = x \circ e^{-1} \circ y$

A heap $(H, [-,-,-])$

$\Downarrow$

$\forall e \in H$, The group $(H, \circ_e, e)$ associated to the heap $(H, [-,-,-])$,

where $x \circ_e y := [x, e, y]$

$\Downarrow$

The heap $(H, [-,-,-]_{\circ_e})$ associated to the group $(H, \circ_e, e)$,

where $[x, y, z]_{\circ_e} := x \circ_e y^{-1} \circ_e z = [[x, e, y^{-1}], e, z] = [[x, e, [e, y, e]], e, z]$




Theorem. Given a group $(G, \circ, 1)$ and $e \in G$,

let $(G, [-,-,-]_\circ)$ be the heap associated to the group $(G, \circ, 1)$,

let $(G, \circ_e, e)$ be the group associated to the heap $(G, [-,-,-]_\circ)$.

Then $(G, \circ, 1) \cong (G, \circ_e, e)$ as groups.

In particular, $\circ = \circ_1$.

Indeed, let $\varphi\colon (G, \circ, 1) \to (G, \circ_e, e)$, $x \mapsto x \circ e$. Then for any $x, y \in G$,

$$\varphi(x \circ y) = (x \circ y) \circ e = (x \circ e) \circ e^{-1} \circ (y \circ e) =$$
$$= \varphi(x) \circ e^{-1} \circ \varphi(y) = [\varphi(x), e, \varphi(y)]_\circ = \varphi(x) \circ_e \varphi(y).$$

Hence φ is a group isomorphism with the inverse $\varphi^{-1}\colon (G, \circ_e, e) \to (G, \circ, 1)$, $x \mapsto x \circ e^{-1}$.

$x \circ y = x \circ 1^{-1} \circ y = [x, 1, y]_\circ = x \circ_1 y$. □




Theorem. Given a heap $(H, [-,-,-])$ and $e \in H$,

let $(H, \circ_e, e)$ be the group associated to the heap $(H, [-,-,-])$,

let $(H, [-,-,-]_{\circ_e})$ be the heap associated to the group $(H, \circ_e, e)$.

Then $[-,-,-] = [-,-,-]_{\circ_e}$.

Indeed, for any $x, y, z \in H$,

$$[x,y,z] = [x,y,[e,e,z]] = [[x,y,e],e,z] =$$

$$= [[[x,e,e],y,e],e,z] = [[x,e,[e,y,e]],e,z] =$$

$$= [[x,e,y^{-1}],e,z] = x \circ_e y^{-1} \circ_e z = [x,y,z]_{\circ_e}.$$

$\square$




Theorem. Let $(H, [-,-,-])$ be a heap.

(a) For any $e, x, y \in H$, if $[e, x, y] = e$ or $[x, y, e] = e$, then $x = y$.

Indeed, since

$e = [e, x, y] = [e, x, y]_{\circ_e} = e \circ_e x^{-1} \circ_e y = x^{-1} \circ_e y$ or

$e = [x, y, e] = [x, y, e]_{\circ_e} = x \circ_e y^{-1} \circ_e e = x \circ_e y^{-1}$,

it follows that $x = y$.

(b) For any $x, y, z, t, u \in H$,

$$[x, [y, z, t], u] = [x, t, [z, y, u]].$$

Indeed, for any $e \in H$,

$$[x, [y, z, t], u] = [x, [y, z, t]_{\circ_e}, u]_{\circ_e} = x \circ_e (y \circ_e z^{-1} \circ_e t)^{-1} \circ_e u =$$

$$= x \circ_e t^{-1} \circ_e z \circ_e y^{-1} \circ_e u = [x, t, [z, y, u]_{\circ_e}]_{\circ_e} = [x, t, [z, y, u]].$$




(c) For any $x, y, z \in H$,

$$[x, y, [y, x, z]] = [x, [y, z, x], y] = [[z, x, y], y, x] = z.$$

Indeed, for any $e \in H$,

$[x, y, [y, x, z]] = [x, y, [y, x, z]_{\circ_e}]_{\circ_e} = x \circ_e y^{-1} \circ_e y \circ_e x^{-1} \circ_e z = z$

$[x, [y, z, x], y] = [x, x, [z, y, y]] = [x, x, z] = z$

$[[z, x, y], y, x] = [[z, x, y]_{\circ_e}, y, x]_{\circ_e} = z \circ_e x^{-1} \circ_e y \circ_e y^{-1} \circ_e x = z.$

(d) If $(H, [-, -, -])$ is an abelian heap, then for any $x_1, x_2, \ldots, z_3 \in H$,

$$[[x_1, x_2, x_3], [y_1, y_2, y_3], [z_1, z_2, z_3]] = [[x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]].$$

Indeed,

$$\begin{aligned}
[[x_1, x_2, x_3], [y_1, y_2, y_3], [z_1, z_2, z_3]] &= [[x_1, x_2, x_3]_{\circ_e}, [y_1, y_2, y_3]_{\circ_e}, [z_1, z_2, z_3]_{\circ_e}]_{\circ_e} = \\
&= x_1 \circ_e x_2{}^{-1} \circ_e x_3 \circ_e (y_1 \circ_e y_2{}^{-1} \circ_e y_3)^{-1} \circ_e z_1 \circ_e z_2{}^{-1} \circ_e z_3 = \\
&= x_1 \circ_e y_1{}^{-1} \circ_e z_1 \circ_e (x_2 \circ_e y_2{}^{-1} \circ_e z_2)^{-1} \circ_e x_3 \circ_e y_3{}^{-1} \circ_e z_3 = \\
&= [[x_1, y_1, z_1]_{\circ_e}, [x_2, y_2, z_2]_{\circ_e}, [x_3, y_3, z_3]_{\circ_e}]_{\circ_e} = [[x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]].
\end{aligned}$$

□

Example. Let $(G, \circ, 1)$ be a group.

Assume that G has more than one element.

Let $H \neq G$ be a subgroup of $(G, \circ, 1)$, and let $x \in G \setminus H$.

Let $(G, [-,-,-]_\circ)$ be the heap associated to the group $(G, \circ, 1)$.

Although the left coset xH is not a subgroup of $(G, \circ, 1)$,

it is a subheap $(xH, [-,-,-]_\circ)$ of $(G, [-,-,-]_\circ)$.

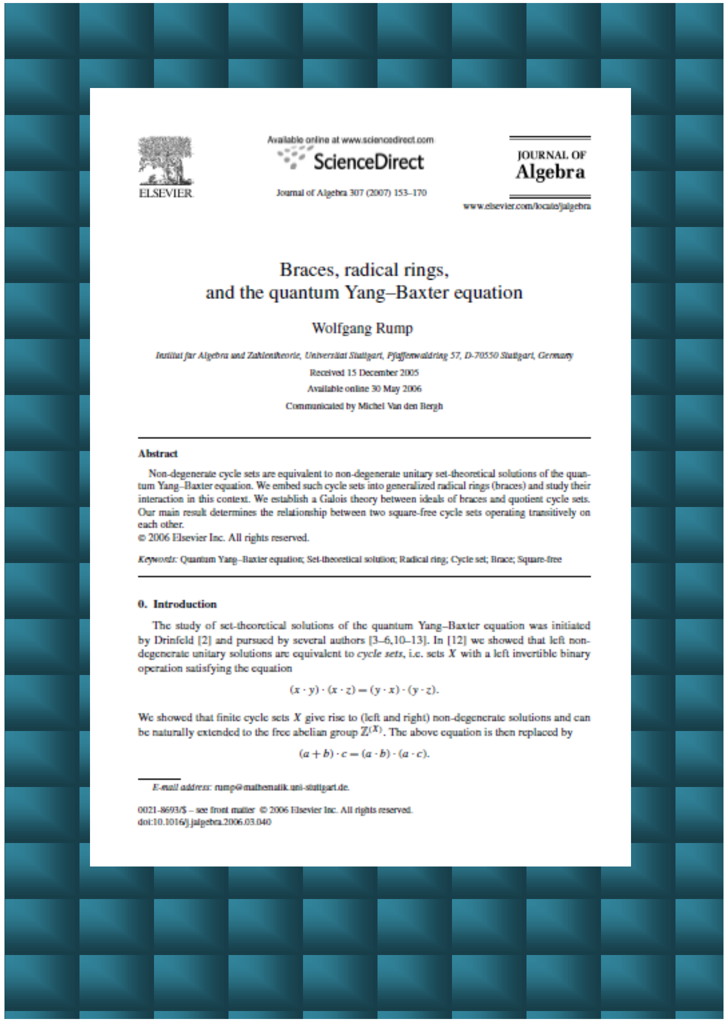
Indeed, for any $a, b, c \in H$,

$$[x \circ a, x \circ b, x \circ c] = x \circ a \circ (x \circ b)^{-1} \circ x \circ c = x \circ a \circ b^{-1} \circ c \in xH. \qquad \square$$

Wolfgang Rump [*B*races, radical rings, and the quantum Yang-Baxter equation, Journal of Algebra 307 (2007) 153-170].

Tomasz Brzeziński [*T*russes: Between braces and rings, Transactions of the American Mathematical Society 372 (2019), 4149–4176].



Available online at www.sciencedirect.com
ScienceDirect
Journal of Algebra 307 (2007) 153–170
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Braces, radical rings, and the quantum Yang–Baxter equation

Wolfgang Rump

Institut für Algebra und Zahlentheorie, Universität Stuttgart, Pfaffenwaldring 57, D-70550 Stuttgart, Germany

Received 15 December 2005
Available online 30 May 2006
Communicated by Michel Van den Bergh

Abstract

Non-degenerate cycle sets are equivalent to non-degenerate unitary set-theoretical solutions of the quantum Yang–Baxter equation. We embed such cycle sets into generalized radical rings (braces) and study their interaction in this context. We establish a Galois theory between ideals of braces and quotient cycle sets. Our main result determines the relationship between two square-free cycle sets operating transitively on each other.

Keywords: Quantum Yang–Baxter equation; Set-theoretical solution; Radical ring; Cycle set; Brace; Square-free

0. Introduction

The study of set-theoretical solutions of the quantum Yang–Baxter equation was initiated by Drinfeld [2] and pursued by several authors [3–6,10–13]. In [12] we showed that left non-degenerate unitary solutions are equivalent to *cycle sets*, i.e. sets X with a left invertible binary operation satisfying the equation

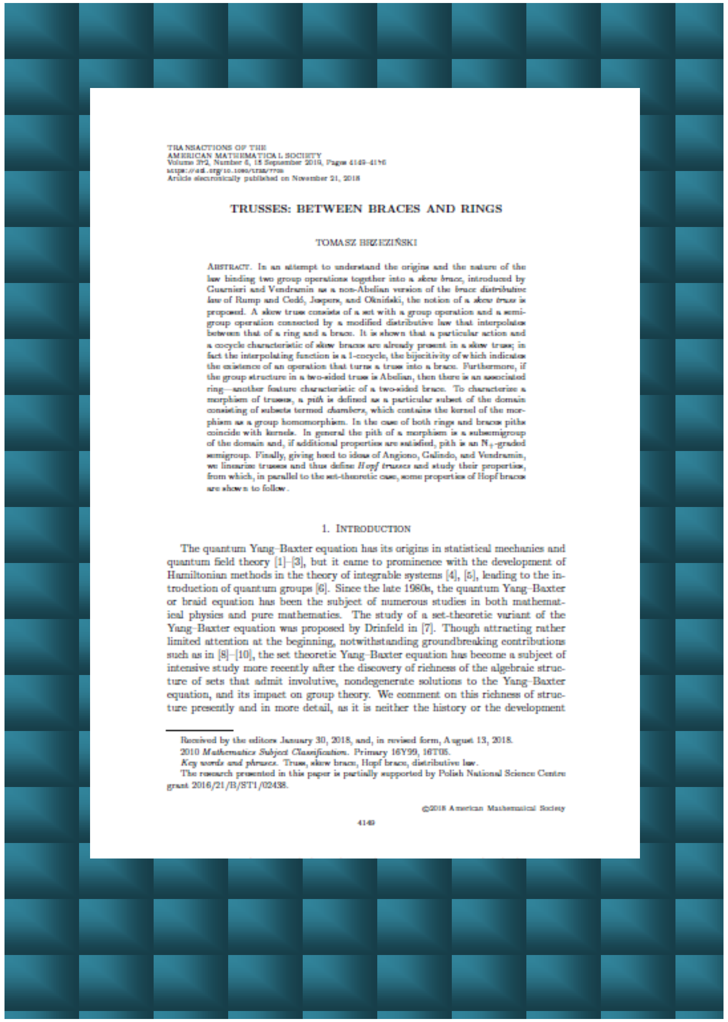
$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z).$$

We showed that finite cycle sets X give rise to (left and right) non-degenerate solutions and can be naturally extended to the free abelian group $\mathbb{Z}^{(X)}$. The above equation is then replaced by

$$(a + b) \cdot c = (a \cdot b) \cdot (a \cdot c).$$

E-mail address: rump@mathematik.uni-stuttgart.de.

0021-8693/$ – see front matter © 2006 Elsevier Inc. All rights reserved.
doi:10.1016/j.jalgebra.2006.03.040



TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 372, Number 6, 15 September 2019, Pages 4149–4176
https://doi.org/10.1090/tran/[illegible]
Article electronically published on November 21, 2018

TRUSSES: BETWEEN BRACES AND RINGS

TOMASZ BRZEZIŃSKI

ABSTRACT. In an attempt to understand the origins and the nature of the law binding two group operations together into a *skew brace*, introduced by Guarnieri and Vendramin as a non-Abelian version of the *brace distributive law* of Rump and Cedó, Jespers, and Okniński, the notion of a *skew truss* is proposed. A skew truss consists of a set with a group operation and a semigroup operation connected by a modified distributive law that interpolates between that of a ring and a brace. It is shown that a particular action and a cocycle characteristic of skew braces are already present in a skew truss; in fact the interpolating function is a 1-cocycle, the bijectivity of which indicates the existence of an operation that turns a truss into a brace. Furthermore, if the group structure in a two-sided truss is Abelian, then there is an associated ring—another feature characteristic of a two-sided brace. To characterize a morphism of trusses, a *pith* is defined as a particular subset of the domain consisting of subsets termed *chambers*, which contains the kernel of the morphism as a group homomorphism. In the case of both rings and braces piths coincide with kernels. In general the pith of a morphism is a subsemigroup of the domain and, if additional properties are satisfied, pith is an $\mathbb{N}_+$-graded semigroup. Finally, giving heed to ideas of Angiono, Galindo, and Vendramin, we linearize trusses and thus define *Hopf trusses* and study their properties, from which, in parallel to the set-theoretic case, some properties of Hopf braces are shown to follow.

1. INTRODUCTION

The quantum Yang–Baxter equation has its origins in statistical mechanics and quantum field theory [1]–[3], but it came to prominence with the development of Hamiltonian methods in the theory of integrable systems [4], [5], leading to the introduction of quantum groups [6]. Since the late 1980s, the quantum Yang–Baxter or braid equation has been the subject of numerous studies in both mathematical physics and pure mathematics. The study of a set-theoretic variant of the Yang–Baxter equation was proposed by Drinfeld in [7]. Though attracting rather limited attention at the beginning, notwithstanding groundbreaking contributions such as in [8]–[10], the set theoretic Yang–Baxter equation has become a subject of intensive study more recently after the discovery of richness of the algebraic structure of sets that admit involutive, nondegenerate solutions to the Yang–Baxter equation, and its impact on group theory. We comment on this richness of structure presently and in more detail, as it is neither the history or the development

Received by the editors January 30, 2018, and, in revised form, August 13, 2018.
2010 *Mathematics Subject Classification.* Primary 16Y99, 16T05.
Key words and phrases. Truss, skew brace, Hopf brace, distributive law.
The research presented in this paper is partially supported by Polish National Science Centre grant 2016/21/B/ST1/02438.

A truss is an algebraic system $(T, [-,-,-], \cdot)$ consisting of a nonempty set T,

a ternary operation

$$[-,-,-]\colon T \times T \times T \to T, \quad (x,y,z) \mapsto [x,y,z],$$

and a binary operation

$$\cdot\colon T \times T \to T, \quad (x,y) \mapsto x \cdot y$$

such that

$(T, [-,-,-])$ is an abelian heap,

$(T, \cdot)$ is a semigroup,

the truss distributivity $\quad x \cdot [y,z,t] = [x \cdot y, x \cdot z, x \cdot t],$

$$[x,y,z] \cdot t = [x \cdot t, y \cdot t, z \cdot t] \text{ holds,}$$

where $x, y, z, t \in T$.

A truss $(T, [-,-,-], \cdot)$ is commutative, if the semigroup $(T, \cdot)$ is commutative.

A truss $(T, [-,-,-], \cdot, 1)$ is unital, if the semigroup $(T, \cdot, 1)$ is a monoid.

A trass homomorphism is a function $\varphi\colon (T, [-,-,-,], \cdot) \to (\tilde{T}, [-,-,-,], \cdot)$

such that

$\varphi\colon (T, [-,-,-,]) \to (\tilde{T}, [-,-,-,])$ is a heap homomorphism,

$\varphi\colon (T, \cdot) \to (\tilde{T}, \cdot)$ is a semigroup homomorphism.

A brace is an algebraic system $(B, +, \cdot, 0, 1)$ consisting of a nonempty set B,

and binary operations

$$+ \colon B \times B \to B, \quad (x, y) \mapsto x + y,$$

$$\cdot \colon B \times B \to B, \quad (x, y) \mapsto x \cdot y$$

such that

$(B, +, 0)$ is an abelian group,

$(B, \cdot, 1)$ is a group,

the brace distributivity $\quad x \cdot (y + z) = x \cdot y - x + x \cdot z,$

$\qquad\qquad (x + y) \cdot z = x \cdot z - z + y \cdot z$ holds,

where $x, y, z \in B$.

Theorem. In a brace $(B, +, \cdot, 0, 1)$, $0 = 1$.

This element will be denoted by θ.

Indeed, from the fact that 0 is the additive identity element

and that 1 is the multiplicative identity element,

it follows that $1 + 0 = 1$ and $0 \cdot 1 = 0$.

Hence

$0 \cdot 1 = 0 \cdot (1 + 0) = 0 \cdot 1 - 0 + 0 \cdot 0 = 0 - 0 + 0 \cdot 0 = 0 \cdot 0$,

and since $\cdot$ is a group operation,

it follows that $1 = 0$. $\square$

Theorem. Given a brace $(B, +, \cdot, \theta)$,

let $(B, [-,-,-,]_+)$ be the abelian heap associated to the abelian group $(B, +, \theta)$.

Then $(B, [-,-,-]_+, \cdot, \theta)$ is a unital truss.

Indeed, for any $x, y, z, t \in B$, since

$x = x \cdot \theta = x \cdot (z - z) = x \cdot z - x + x \cdot (-z)$,

it follows that

$x \cdot (-z) = x - x \cdot z + x$,

and thus

$$
\begin{aligned}
x \cdot [y, z, t]_+ &= x \cdot (y - z + t) = x \cdot y - x + x \cdot (-z) - x + x \cdot t = \\
&= x \cdot y - x + (x - x \cdot z + x) - x + x \cdot t = \\
&= x \cdot y - x \cdot z + x \cdot t = [x \cdot y, x \cdot z, x \cdot t]_+ .
\end{aligned}
$$

In a similar manner, $[x, y, z]_+ \cdot t = [x \cdot t, y \cdot t, z \cdot t]_+$. □






Theorem. Given a ring $(R,+,\cdot,0)$,

let $(R,[-,-,-,]_+)$ be the abelian heap associated to the abelian group $(R,+,0)$.

Then $(R,[-,-,-]_+,\cdot,0)$ is a truss.

Indeed, for any $x,y,z,t\in R$,

$x\cdot[y,z,t]_+ = x\cdot(y-z+t) = x\cdot y - x\cdot z + x\cdot t = [x\cdot y, x\cdot z, x\cdot t]_+$.

In a similar manner, $[x,y,z]_+\cdot t = [x\cdot t, y\cdot t, z\cdot t]_+$.

$\square$

Theorem. Given a truss $(T, [-,-,-], \cdot)$ and $e \in T$,

let $(T, +_e, e)$ be the abelian group associated to the abelian heap $(T, [-,-,-])$.

Then $(T, +_e, \cdot, e)$ is a ring iff

$$e \cdot x = e = x \cdot e \quad \text{for any } x \in T.$$

An element $e \in T$ with this property is called an absorber.

If an absorber exists, then it is unique.

Indeed, if $(T, +_e, \cdot, e)$ is a ring, then since e is the zero element,

it follows that $e \cdot x = e = x \cdot e$ for any $x \in T$.

If e is an absorber in the truss $(T, [-,-,-], \cdot)$, then for any $x, y, z \in T$,

$x \cdot (y +_e z) = x \cdot [y, e, z] = [z \cdot y, x \cdot e, x \cdot z] = [z \cdot y, e, x \cdot z] = x \cdot y +_e x \cdot z$.

In a similar manner, $(x +_e y) \cdot z = x \cdot z +_e y \cdot z$.

If e, f are absorbers in the truss $(T, [-,-,-], \cdot)$, then $e = e \cdot f = f$. $\square$






Example. Let $(H, [-,-,-])$ be an abelian heap.

Assume that H has more than one element. Let

$$\cdot\colon H \times H \to H, \qquad x \cdot y := x,$$

where $x, y \in H$. Then

(a) $(H, [-,-,-], \cdot)$ is a noncommutative truss.

Indeed, for any $x, y, z, t \in H$,

$x \cdot (y \cdot z) = x = x \cdot z = (x \cdot y) \cdot z$

$x \cdot [y, z, t] = x = [x, x, x] = [x \cdot y, x \cdot z, x \cdot t]$

$[x, y, z] \cdot t = [x, y, z] = [x \cdot t, y \cdot t.z \cdot t]$

$x \cdot y = x \neq y = y \cdot x$ as long as $x \neq y$.




(b) $(H, [-,-,-], \cdot)$ is nonunital,

and hence $(H, [-,-,-], \cdot)$ is not arrising from any brace.

Indeed, if the truss $(H, [-,-,-], \cdot)$ was unital,

the operation $\cdot$ would need to have the identity element, say 1.

But for any $x \in H$, if $x \neq 1$ then $1 \cdot x = 1 \neq x$.

Thus 1 cannot be the identity element.

(c) $(H, [-,-,-], \cdot)$ has no absorbers,

and hence $(H, [-,-,-], \cdot)$ is not arrising from any ring.

Indeed, if the truss $(H, [-,-,-], \cdot)$ was arrising from a ring,

it would need to have the absorber, say 0.

But for any $x \in H$, if $x \neq 0$ then $x \cdot 0 = x \neq 0$.

Thus 0 cannot be the absorber. □




Example. Let $(\mathbb{Z}, +, \cdot, 0, 1)$ be the ring of integer numbers,

and let $(\mathbb{Z}, [-,-,-]_+, \cdot)$ be the truss associated to the ring $(\mathbb{Z}, +, \cdot, 0, 1)$.

Although the set of odd integer numbers $2\mathbb{Z}+1$ is not a subring of $(\mathbb{Z}, +, \cdot, 0, 1)$,

it is a subtruss $(2\mathbb{Z}+1, [-,-,-]_+, \cdot)$ of $(\mathbb{Z}, [-,-,-]_+, \cdot)$.

$\square$

Theorem. Let $(H, [-,-,-])$ be a heap, and let $S \subseteq H$ be a nonempty subset.

Then the following statements are equivalent

(a) S is a subheap $(S, [-,-,-])$ of $(H, [-,-,-])$.

(b) For every $e \in S$,

S is a subgroup $(S, \circ_e, e)$ of the associated group $(H, \circ_e, e)$.

Indeed, for (a)⇒(b), let $s, t \in S$. Then

$s \circ_e t^{-1} = [s, e, [e, t, e]] = [[s, e, e], t, e] = [s, t, e] \in S$.

(c) For some $e \in S$,

S is a subgroup $(S, \circ_e, e)$ of the associated group $(H, \circ_e, e)$.

Indeed, for (c)⇒(a), let $s, t, u \in S$. Then

$[s, t, u] = [s, t, u]_{\circ_e} = s \circ_e t^{-1} \circ_e u \in S$. □

A subheap S of a heap $(H, [-,-,-])$ is normal, if

$$\exists e \in S \ \forall s \in S \ \forall x \in H \ \exists t \in S, \ \ [x, e, s] = [t, e, x].$$

Theorem. Let $(H, [-,-,-])$ be a heap, and let $S \subseteq H$ be a nonempty subset.

Then the following statements are equivalent

(a) S is a normal subheap $(S, [-,-,-])$ of $(H, [-,-,-])$.

(b) $\forall e, s \in S \ \forall x \in H \ \exists t \in S, \ \ [x, e, s] = [t, e, x]$.

Indeed, for (a)$\Rightarrow$(b), let $e \in S$ be such that

$\forall s \in S \ \forall x \in H \ \exists t \in S, \ \ [x, e, s] = [t, e, x]$.

Then for any $f, s \in S$, $x \in H$ and for some $u \in S$, since $[x, e, [e, f, s]] = [u, e, x]$,

it follows that $[[x, e, e], f, s] = [u, e, [f, f, x]]$,

and thus $[x, f, s] = [[u, e, f], f, x]$, where $[u, e, f] \in S$.






(c) $\forall e, s \in S \ \forall x \in H, \ [[x, e, s], x, e] \in S.$

Indeed, for (b)⇒(c), let $e, s \in S$, $x \in H$,

and let $t \in S$ be such that $[x, e, s] = [t, e, x]$. Then

$[[x, e, s], x, e] = [[t, e, x], x, e] = [t, e, [x, x, e]] = [t, e, e] = t \in S.$

For (c)⇒(a), let $e, s \in S$ $x \in H$,

and let $t \in S$ be such that $[[x, e, s], x, e] = t$.

Then since $[[[x, e, s], x, e], e, x] = [t, e, x]$,

it follows that $[x, e, s] = [t, e, x]$.

(d) For every $e \in S$,

S is a normal subgroup $(S, \circ_e, e)$ of the associated group $(H, \circ_e, e)$.

Indeed, for (b)⇒(d), let $e, s \in S$, $x \in H$,

and let $t \in S$ be such that $[x, e, s] = [t, e, x]$.

Then since $x \circ_e s = t \circ_e x$,

it follows that $x \circ_e s \circ_e x^{-1} = t \in S$.

(e) For some $e \in S$,

S is a normal subgroup $(S, \circ_e, e)$ of the associated group $(H, \circ_e, e)$.

Indeed, for (e)⇒(a), let $s \in S$, $x \in H$,

and let $t \in S$ be such that $x \circ_e s \circ_e x^{-1} = t$.

Then since $x \circ_e s = t \circ_e x$,

it follows that $[x, e, s] = [t, e, x]$. □

Given a subheap S of a heap $(H, [-,-,-])$,

let the subheap relation $\sim_S$ on H be defined as

$$x \sim_S y \quad :\Leftrightarrow \quad \text{for some } s \in S, [x,y,s] \in S.$$

In the associated group $(H, \circ_s, s)$, this means that

$$x \circ_s y^{-1} = x \circ_s y^{-1} \circ_s s = [x,y,s]_{\circ_s} = [x,y,s] \in S.$$

Theorem. Let S be a subheap $(S, [-,-,-])$ of a heap $(H, [-,-,-])$. Then

(a) $x \sim_S y \quad \Leftrightarrow \quad$ for every $s \in S, [x,y,s] \in S$.

Indeed, let $x \sim_S y$, and let $s \in S$ be such that $[x,y,s] \in S$.

Then for any $t \in S$,

$[x,y,t] = [x,y,[s,s,t]] = [[x,y,s],s,t] \in S$.






(b) $\sim_S$ is an equivalence relation on $(H, [-,-,-])$.

Indeed, for any $x, y, z \in H$, $s \in S$,

since $[x, x, s] = s \in S$, it follows that $x \sim_S x$.

If $x \sim_S y$, then since $[x, y, s] \in S$, it follows that $[y, x, [x, y, s]] = s \in S$, and thus $y \sim_S x$.

If $x \sim_S y$ and $y \sim_S z$, then since $[x, y, S] \subseteq S$ and $[y, z, s] \in S$, it follows that

$[x, z, s] = [[x, y, y], z, s] = [x, y, [y, z, s]] \in S$, and thus $x \sim_S z$.

The equivalence class with respect to the subheap relation $\sim_S$ will be denoted by

$$\overline{x} := \left\{x' \in H \mid \text{for some } s \in S,\ [x', x, s] \in S\right\} = \\ = \left\{x' \in H \mid \text{for every } s \in S,\ [x', x, s] \in S\right\}.$$

(c) For every $s \in S$, $\overline{s} = S$.

Indeed, for any $x \in H$, from the fact that $[x, s, s] = x$,

it follows that $x \in \overline{s}$ if and only if $x \in S$.

(d) For every $x \in H$, $\overline{x}$ is a subheap $(\overline{x}, [-,-,-])$ of $(H, [-,-,-])$.

Indeed, for any $y, z, t \in \overline{x}$, $s \in S$, since $y \in \overline{z}$,

it follows that $[[y,z,t],x,s] = [y,z,[t,x,s]] \in S$,

and thus $[y,z,t] \in \overline{x}$.

(e) For any $x, y \in H$, the function

$$\tau_y^x \colon (H, [-,-,-]) \to (H, [-,-,-]), \quad \tau_y^x(z) := [z,y,x]$$

is a heap automorphism with the inverse $(\tau_y^x)^{-1} = \tau_x^y$.

Indeed, for any $z, t, u \in H$,

$$[\tau_y^x(z), \tau_y^x(t), \tau_y^x(u)] = [[z,y,x], \tau_y^x(t), \tau_y^x(u)] = [z,y,[x,[t,y,x],\tau_y^x(u)]] =$$

$$= [z,y,[x,x,[y,t,\tau_y^x(u)]]] = [z,y,[y,t,\tau_y^x(u)]] =$$

$$= [[z,y,y],t,\tau_y^x(u)] = [z,t,[u,y,x]] = [[z,t,u],y,x] = \tau_y^x([z,t,u]).$$






(f) For any $x, y \in H$, $\overline{x} = \tau_y^x(\overline{y})$.

Indeed, if $x' \in \overline{x}$, then since

$$x' = (\tau_y^x \circ \tau_x^y)(x') = \tau_y^x([x', x, y]),$$

and since for any $s \in S$,

$$[[x', x, y], y, s] = [x', x, [y, y, s] = [x', x, s]] \in S,$$

it follows that $x' = \tau_y^x([x', x, y]) \in \tau_y^x(\overline{y})$.

If $y' \in \overline{y}$, then for any $s \in S$,

$$[\tau_y^x(y'), x, s] = [[y', y, x]x, s] = [y', y, [x, x, s]] = [y', y, s] \in S,$$

and thus $\tau_y^x(\overline{y}) \in \overline{x}$.

(g) For any $x, y \in H$, $(\overline{x}, [-,-,-]) \cong (\overline{y}, [-,-,-])$ as heaps. $\square$

Theorem. Let S be a normal subheap $(S, [-,-,-])$ of a heap $(H, [-,-,-])$.

Then

(a) $\sim_S$ is a congruence in $(H, [-,-,-])$, that is,

$\sim_S$ is an equivalence relation on $(H, [-,-,-])$ and

$x \sim_S x'$, $y \sim_S y'$, $z \sim_S x'$ imply that $[x, y, z] \sim_S [x', y', z']$,

where $x, x', \ldots, z' \in H$.

Indeed, if $x \sim_S x'$, $y \sim_S y'$, $z \sim_S x'$, then for any $e \in S$,

$x' \in \overline{x} = \tau_e^x(\overline{e}) = \tau_e^x(S)$

$y' \in \overline{y} = \tau_e^y(\overline{e}) = \tau_e^y(S)$

$z' \in \overline{z} = \tau_e^z(\overline{e}) = \tau_e^z(S)$.

In the associated group $(H, \circ_e, e)$, this means that

$\overline{x} = \tau_e^x(s) = [s, e, x] = s \circ_e x$

$\overline{y} = \tau_e^y(t) = [t, e, y] = t \circ_e y$

$\overline{z} = \tau_e^z(u) = [u, e, z] = z \circ_e z$

for some $s, t, u \in S$. Then

$$[x', y', z'] = [s \circ_e x, t \circ_e y, u \circ_e z]_{\circ_e} = s \circ_e x \circ_e (t \circ_e y)^{-1} \circ_e u \circ_e z =$$
$$= s \circ_e x \circ_e y^{-1} \circ_e t^{-1} \circ_e u \circ_e y \circ_e x^{-1} \circ_e x \circ_e y^{-1} \circ_e z =$$
$$= v \circ_e [x, y, z]_{\circ_e} = [v, e, [x, y, z]],$$

where $v = s \circ_e x \circ_e y^{-1} \circ_e t^{-1} \circ_e u \circ_e y \circ_e x^{-1}$.

From the fact that S is a normal subgroup $(S, \circ_e, e)$ of $(H, \circ_e, e)$, it follows that $v \in S$.

Hence

$$[x', y', z'] = [v, e, [x, y, z]] = \tau_e^{[x,y,z]}(v) \in \tau_e^{[x,y,z]}(S) = \tau_e^{[x,y,z]}(\overline{e}) = \overline{[x, y, z]},$$

and thus $[x, y, z] \sim_S [x', y', z']$.

(b) the set of equivalence classes $H/\sim_S$ is a heap with the ternary operation

$$[-,-,-] \colon H/\sim_S \times H/\sim_S \times H/\sim_S \to H/\sim_S, \quad [\overline{x}, \overline{y}, \overline{z}] := \overline{[x, y, z]},$$

where $x, y, z \in H$.

Indeed, if $\overline{x} = \overline{x'}$, $\overline{y} = \overline{y'}$, $\overline{z} = \overline{z'}$, then since $x \sim_S x'$, $y \sim_S y'$, $z \sim_S z'$,

it follows that $[x, y, z] \sim_S [x, y, z]$, and thus $\overline{[x, y, z]} = \overline{[x', y', z']}$. $\square$

Given a heap homomorphism $\varphi\colon (H, [-,-,-]) \to (\widetilde{H}, [-,-,-])$,

let the kernel relation of φ on H be defined as

$$x \; Ker\varphi \; y \quad :\Leftrightarrow \quad \varphi(x) = \varphi(y).$$

Theorem. Let $(H, [-,-,-])$ be a heap.

(a) For any heap homomorphism $\varphi\colon (H, [-,-,-]) \to (\widetilde{H}, [-,-,-])$,

$Ker\varphi$ is a congruence in $(H, [-,-,-])$.

Indeed, if $x \; Ker\varphi \; x'$, $y \; Ker\varphi \; y'$, $z \; Ker\varphi \; z'$,

then since $\varphi(x) = \varphi(x')$, $\varphi(y) = \varphi(y')$, $\varphi(z) = \varphi(z')$,

it follows that

$\varphi([x,y,z]) = [\varphi(x), \varphi(y), \varphi(z)] = [\varphi(x'), \varphi(y'), \varphi(z')] = \varphi([x',y',z'])$,

and thus $[x,y,z] \; Ker\varphi \; [x',y',z']$.






(b) If ρ is a congruence in $(H,[-,-,-])$, then

(i) the set of equivalence classes H/ρ is a heap with the ternary operation

$$[-,-,-]\colon H/\rho\times H/\rho\times H/\rho\to H/\rho,\qquad [\widehat{x},\widehat{y},\widehat{z}]:=[\widehat{x,y,z}],$$

where $x,y,z\in H$, and $\widehat{x}$, $\widehat{y}$, $\widehat{z}$ mean the equivalence classes with respect to the relation ρ.

(ii) the function

$$\varphi\colon (H,[-,-,-])\to(\widetilde{H},[-,-,-]),\qquad x\mapsto\widehat{x}$$

is a heap homomorphism such that $\rho=Ker\varphi$.

Indeed, for any $x,y,z\in H$,

$$\varphi([x,y,z])=[\widehat{x,y,z}]=[\widehat{x},\widehat{y},\widehat{z}]=[\varphi(x),\varphi(y),\varphi(z)]$$

$$x\ \rho\ y\quad\Leftrightarrow\quad\widehat{x}=\widehat{y}\quad\Leftrightarrow\quad\varphi(x)=\varphi(y)\quad\Leftrightarrow\quad x\ Ker\varphi\ y.$$ □




The equivalence class with respect to the kernel relation $Ker\varphi$ will be denoted by

$$\widehat{x} := \left\{x' \in H \mid \varphi(x) = \varphi(x')\right\}.$$

Theorem. Under the above notations, for every $x \in H$,

$\widehat{x}$ is a normal subheap $(\widehat{x}, [-,-,-])$ of $(H, [-,-,-])$.

Indeed, for any $s, t, u \in \widehat{x}$, $y \in H$, since

$\varphi([s,t,u]) = [\varphi(s), \varphi(t), \varphi(u)] = [\varphi(x), \varphi(x), \varphi(x)] = \varphi(x)$,

it follows that $[s,t,u] \in \widehat{x}$.

Since

$$\begin{aligned}\varphi([[y,s,t],ys]) &= [[\varphi(y), \varphi(s), \varphi(t)], \varphi(y), \varphi(s)] = \\ &= [[\varphi(y), \varphi(x), \varphi(x)], \varphi(y), \varphi(x)] = [\varphi(y), \varphi(y), \varphi(x)] = \varphi(x),\end{aligned}$$

it follows that $[[y,s,t],y,s] \in \widehat{x}$,

and thus $\widehat{x}$ is a normal subheap $(\widehat{x}, [-,-,-])$ of $(H, [-,-,-])$. □






Given a heap homomorphism $\varphi\colon (H, [-,-,-]) \to (\widetilde{H}, [-,-,-])$, and $e \in Im\varphi$,

the e-kernel of φ is the subset of H defined as

$$ker_e\varphi := \Big\{x \in H \mid \varphi(x) = e\Big\}.$$

Theorem. Under the above notations,

(a) for every $e \in Im\varphi$,

$ker_e\varphi$ is a normal subheap $(ker_e\varphi, [-,-,-])$ of $(H, [-,-,-])$.

Indeed, for any $x \in ker_e\varphi$, since

$$y \in ker_e\varphi \quad \Leftrightarrow \quad \varphi(y) = e \quad \Leftrightarrow \quad \varphi(y) = \varphi(x) \quad \Leftrightarrow \quad x\ Ker\varphi\ y \quad \Leftrightarrow \quad y \in \widehat{x},$$

it follows that

$ker_e\varphi = \widehat{x}$, the equivalence class with respect the kernel relation $Ker\varphi$.

Thus

$ker_e\varphi$ is a normal subheap $(ker_e\varphi, [-,-,-])$ of $(H, [-,-,-])$.

(b) For any $e, f \in Im\varphi$, $(ker_e\varphi, [-,-,-]) \cong (ker_f\varphi, [-,-,-])$ as heaps.

Indeed, let $x \in ker_e\varphi$, $y \in ker_f\varphi$.

It suffices to prove that $ker_e\varphi = \tau_y^x(ker_f\varphi)$.

If $z \in ker_e\varphi$, then since

$z = (\tau_y^x \circ \tau_x^y)(z) = \tau_y^x([z, x, y])$,

and since

$\varphi([z, x, y]) = [\varphi(z), \varphi(x), \varphi(y)] = [e, e, f] = f$,

it follows that

$z = \tau_y^x([z, x, y]) \in \tau_y^x(ker_e\varphi)$.

If $z \in ker_f\varphi$, then

$\varphi(\tau_y^x(z)) = \varphi([z, y, x]) = [\varphi(z), \varphi(y), \varphi(x)] = [f, f, e] = e$,

and thus $\tau_y^x(z) \in ker_e\varphi$.






(c) For ane $e \in Im\varphi$, $\sim_{ker_e\varphi} = Ker\varphi$.

Indeed, if $x \sim_{ker_e\varphi} y$, then for any $s \in ker_e\varphi$,

$[x, y, s] \in ker_e\varphi$ by the definition of the subheap relation $\sim_{ker_e\varphi}$.

From this it follows that

$e = \varphi([y, x, s]) = [\varphi(x), \varphi(y), \varphi(s)] = [\varphi(x), \varphi(y), e]$,

which means that $\varphi(x) = \varphi(y)$, and thus

$x\ Ker\varphi\ y$ by the definition of the kernel relation $Ker\varphi$.

If $x\ Ker\varphi\ y$, then since $\varphi(x) = \varphi(y)$,

it follows that for any $s \in ker_e\varphi$,

$\varphi([x, y, s]) = [\varphi(x), \varphi(y), \varphi(s)] = [\varphi(x), \varphi(x), e] = e$,

which means that $[x, y, s] \in ker_e\varphi$,

and thus $x \sim_{ker_e\varphi} y$. □






Corollary. Let $(H,[-,-,-])$ be a heap,

and let ρ be an equivalence relation on $(H,[-,-,-])$.

Then the following statemants are equivalent

(a) ρ is a congruence in $(H,[-,-,-])$.

(b) There exists a heap homomorphism $\varphi:(H,[-,-,-])\to(\widetilde{H},[-,-,-])$

such that $\rho = Ker\varphi$.

(c) There exists a normal subheap $(S,[-,-,-])$ of $(H,[-,-,-])$

such that $\rho = \sim_S$. $\square$




Let $(T, [-,-,-], \cdot)$ be a truss.

A subheap $(S, [-,-,-])$ of the heap $(T, [-,-,-])$

is an ideal of the truss $(T, [-,-,-], \cdot)$, if

$$s \cdot x \in S, \quad x \cdot s \in S,$$

where $s \in S$, $x \in T$.

Theorem. Let S be an ideal of a truss $(T, [-,-,-], \cdot)$. Then

$\sim_S$ is a congruence in the truss $(T, [-,-,-], \cdot)$, that is,

$\sim_S$ is a congruence in the heap $(T, [-,-,-])$,

$x \sim_S x'$, $y \sim_S y'$ imply that $x \cdot y \sim_S x' \cdot y'$,

where $x, x', y, y' \in H$. $\square$

Theorem. Let $\varphi\colon (T,[-,-,-],\cdot) \to (\tilde{T},[-,-,-],\cdot)$ be a truss homomorphism, and let $e \in Im\varphi$. Then

(a) For any $s \in ker_e\varphi$, $x \in T$, $s \cdot x \in ker_e\varphi \quad \Leftrightarrow \quad$ for any $y \in Im\varphi$, $e \cdot y = e$.

(b) For any $s \in ker_e\varphi$, $x \in T$, $x \cdot s \in ker_e\varphi \quad \Leftrightarrow \quad$ for any $y \in Im\varphi$, $y \cdot e = e$.

$\square$

Theorem. Let $\varphi\colon (T,[-,-,-],\cdot) \to (\widetilde{T},[-,-,-],\cdot)$ be a truss homomorphism, and let $e \in Im\varphi$. Then for any $p,q \in ker_e\varphi$, $x \in T$,

$$[x\cdot p, x\cdot q, q] \in ker_e\varphi, \qquad [p\cdot x, q\cdot x, q] \in ker_e\varphi.$$

$\square$

Let $(T,[-,-,-],\cdot)$ be a truss.

A subheap $(P,[-,-,-])$ of the heap $(T,[-,-,-])$

is a paragon of the truss $(T,[-,-,-],\cdot)$, if

$$[x\cdot p, x\cdot q, q] \in P, \qquad [p\cdot x, q\cdot x, q] \in P,$$

where $p,q \in P$, $x \in T$.




Example. Let $(R, +, \cdot, 0)$ be a ring.

Assume that R has more than one element.

Let $I \neq R$ be an ideal of $(R, +, \cdot, 0)$, and let $x \in T \setminus I$.

Let $(R, [-,-,-]_+, \cdot)$ be the trass associated to the ring $(R, +, \cdot, 0)$.

Although the coset $x + I$ is not an ideal of $(R, +, \cdot, 0)$,

it is a paragon of $(R, [-,-,-]_+, \cdot)$. $\square$

Theorem. Let $(T, [-,-,-], \cdot)$ be a truss,

and let ρ be an equivalence relation on $(T, [-,-,-], \cdot)$.

Then the following statemants are equivalent

(a) ρ is a congruence in $(T, [-,-,-], \cdot)$.

(b) There exists a truss homomorphism $\varphi : (T, [-,-,-], \cdot) \to (\tilde{T}, [-,-,-], \cdot)$

such that $\rho = Ker\varphi$.

(c) There exists a paragon P of $(T, [-,-,-], \cdot)$

such that $\rho = \sim_P$. $\square$




Thank you very much for your attention!

Merci beaucoup pour votre attention!